MIXED FINITE ELEMENT METHODS – REDUCED AND SELECTIVE INTEGRATION TECHNIQUES: A UNIFICATION OF CONCEPTS

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The equivalence of certain classes of mixed finite element methods with displacement methods which employ reduced and selective integration techniques is established. This enables one to obtain the accuracy of the mixed formulation without incurring the additional computational expense engendered by the auxiliary field of the mixed method. Applications and numerical examples are presented for problems with constraints which can be difficult to enforce in finite element approximations and have often dictated the use of mixed principles. These include thin beams and plates, and linear and nonlinear incompressible and nearly incompressible continuum problems in solid and fluid mechanics.

1. Introduction

Mixed finite element formulations, in the context of a variational theorem, were first discussed by Fraeijs de Veubeke [1] and Herrmann [2]. Herrmann developed a reduced form of Reissner’s principle particularly suited to problems of incompressible and nearly incompressible elasticity and, based upon this principle, established the first effective finite elements for such cases. Prior to this development many displacement models, derived from the single-field potential energy principle, were applied to these problems, and each was a resounding failure. The reasons for this were not understood. Certain elements derived from Herrmann’s principle also failed. Hughes and Allik [3] traced this failure to a correspondence between mixed and displacement models contained within Fraeijs de Veubeke’s “limitation principle” [1]. Mixed methods have subsequently been applied to many problems in engineering and have been especially successful in “constrained problems”, e.g. incompressible and nearly incompressible elasticity and the bending of plates and shells. Lately, a very complete mathematical theory has been developed for mixed methods in linear analysis (see Babuska, Oden and Lee [4]). However, there are at least two disadvantages.

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First, the auxiliary field of the mixed method engenders additional computational expense, and, second, the generalization to nonlinear problems is not always apparent.

The first example of a reduced integration element was apparently the plate/shell element presented by Zienkiewicz, Taylor and Too [5]. Two-by-two Gauss quadrature was employed on the 8-node serendipity element, and considerable improvement over the $3 \times 3$ Gauss rule was noted. The same concept was employed in other areas by Zienkiewicz and his colleagues. In particular, Naylor [6] and Zienkiewicz and Godbole [7] advocated the use of the 8-node serendipity element in problems involving incompressibility. The procedure, however, was viewed by many as more a trick than a method, and subsequently some bad experiences were noted for the serendipity element.

The concept of selective integration was first employed by Doherty, Wilson and Taylor [8] to obtain improved bending behavior in simple 4-node elasticity elements. One-point Gauss quadrature was used on the shear strain term, and $2 \times 2$ Gauss quadrature was used to integrate the remaining terms. Although improved behavior was noted in some configurations, lack of invariance caused the approach to be abandoned.

Studies performed by Fried [9], Nagtegaal, Parks and Rice [10], and Argyris, Dunne, Angelopoulos and Bichat [11] provided fresh insights into why the displacement approach failed in constrained problems. It was Fried who suggested reduced/selective integration as a possible cure, and considerable practical and theoretical progress has been made since then. Malkus [12, 13] proved the equivalence of a class of mixed models with reduced/selective integration single-field elements in linear elasticity theory. Malkus also applied the reduced/selective integration technique to problems in rheology [14, 15]. Hughes and co-workers applied the concept to linear and nonlinear elasticity [16, 17], the Navier-Stokes equations [18, 19] and plate bending [20]. Motivated by efficiency considerations, the simplest reduced/selective Lagrange element was advocated by Hughes, who also proved [21] the equivalence of this element with one previously derived from Herrmann's principle [2].

Recent investigations of note include Argyris, Dunne, Johnsen and Müller [22], in which the computational and programming aspects of constrained media situations are investigated, and Sandhu and Singh [23], in which a solution strategy is proposed for a certain rank-deficient, reduced integration element.

In this paper we present new, more general equivalent results for mixed methods and displacement methods employing the reduced/selective integration technique. Theorems are stated and proved for both linear and nonlinear problems. Applications and numerical examples are given for beams, plates and incompressible and nearly incompressible continuum problems in fluid and solid mechanics.

We believe the practical computing consequences of these results are significant. Namely, the accuracy of the mixed formulation in constrained media situations can be obtained with a displacement formulation, completely eliminating the additional computational expense engendered by the auxiliary field of the mixed formulation. Furthermore, application to nonlinear problems is straightforward and efficient.

The results also have theoretical significance in that the convergence proofs and error estimates developed for the mixed methods [4] immediately apply to the reduced/selective integration displacement elements. Hence, we elevate the latter approach from the realm of tricks to a legitimate methodology.
2. Linear analysis

2.1. Equivalence of the solutions

We consider mixed variational principles which have as an Euler equation a compatibility relation between a strain-related functional of the displacements and an auxiliary field. Let \( u_i \) denote the displacements or velocities, let \( g \) be the strain-related auxiliary field, and let \( L \) be an operator which assigns to \( u_i \) the compatible strain-related variable \( \epsilon_i \). The Euler equation under consideration takes the form

\[
L(u_i) - \alpha g = 0 ,
\]

where \( \alpha > 0 \) is a constant. The Euler equation comes from the variation of the term \( \Psi \) in the functional, viz.

\[
\Psi = gL(u_i) - \frac{\alpha}{2} g^2 ,
\]

\[
\frac{\delta \Psi}{\delta g} = L(u_i) - \alpha g .
\]

The vector and tensor generalizations follow easily from the scalar case.

Example 1. Herrmann incompressibility principle:

\[
L(u_i) = e_{11} + e_{22} + e_{33} = \text{volume strain},
\]

\( g = H = \text{mean pressure variable}, \)

\( \alpha = 1 - 2\nu, \quad \nu = \text{Poisson's ratio}. \)

Example 2. Beam (rectangular cross-section):

\[
L(u_i) = w_{xy} - \theta = \text{shear strain},
\]

\( g = q = \text{shear force}, \)

\( \alpha = \frac{t^2(1 + \nu)}{6}, \quad t = \text{thickness}. \)

Example 3. Plate:

\[
L(u_i) = \begin{bmatrix} w_{11} - \theta_1 \\ w_{22} - \theta_2 \end{bmatrix} = \text{shear strains},
\]
Example 4. Mixed principle for the Laplacian \( (Oden \text{ and } Reddy \ [24]) \):

\[
L(u_i) = \begin{bmatrix} u_{1} \\ u_{2} \end{bmatrix},
\]

\[
g = \begin{bmatrix} v_{1} \\ v_{2} \end{bmatrix},
\]

\[
\alpha = 1.
\]

Remarks 1. It is usual in engineering practice to include “shear correction factors” in the relation between shear forces and shear strains in beam and plate theories. For simplicity we have assumed these factors have value unity.

Remark 2. Single-field principles can be obtained from mixed principles of the above type by solving (1) on the space of all admissible functions a priori and substituting \( L(u_i)/\alpha \) for \( g \) in \( \Psi \).

Remark 3. In finite element applications the situation is different. The Euler equations are only satisfied in the sense

\[
\int_{\Omega} \phi_k \frac{\partial \Psi}{\partial g} \, d\Omega = 0, \quad \forall \phi_k \in T^h,
\]

where \( T^h \) is the trial space of the \( g \)'s. This implies \( \partial \Psi/\partial g = 0 \) only if \( L(u_i) \in T^h \), which is Fraeijs de Veubeke’s limitation principle [1]. It is well known that the use of mixed finite element methods provides no advantage when the limitation principle holds. However, we will show that by using numerical integration and constructing trial spaces with \( L(u_i) \notin T^h \) the Euler equation can be solved at the integration points, giving a single-field principle retaining the advantages of the mixed principle.

The following construction is crucial to the equivalent theorem:

\[ (A) \]

Given an integration formula for each element, construct a trial space \( T^h \) with the integration points for nodes.

Construct the element assemblage without interelement continuity of the \( g \)'s (separate nodal values on opposite sides of boundaries if necessary).

Remark 4. Such a \( T^h \) has a nodal interpolate even to functions with interelement discontinuities.

Theorem 1. The following two procedures produce the same finite element solution:
(mixed method)
1) Construct $T_h$ as in (A).
2) Using a mixed principle on the trial space $(S^h, T^h)$, $u_i \in S^h, g \in T^h$, evaluate the element matrices associated with $\Psi$ using the integration formula based on the nodes of $T^h$.
3) Use some sufficiently accurate integration scheme on any remaining terms.
4) Obtain the stationary point on $(S^h, T^h)$ with nodal values $(u, g)$.

(reduced/selective integration)
1' Perform the substitution of remark 2 above, obtaining a single-field principle.
2' Evaluate the element matrices of the new $\Psi$ with the integration points which would have been nodes of $T^h$.
3' Use the same formula as in 3) of the mixed method for any remaining terms.
4' Obtain the stationary point on $S^h$ and generate $g \in T^h$ by taking for nodal values of $g$, $g = \{a^i L(u_i)(\xi_i)\}$, where $\xi_i$ are the integration points, giving a solution in $(S^h, T^h)$, $(u, g)$.

Proof. The relation

$$
\int_{\Omega} \frac{\partial \Psi}{\partial g} \, d\Omega + \sum_e \sum_l W^e_i \phi^e_k(\xi^e_i) \frac{\partial \Psi}{\partial g}(\xi^e_i) = 0 ,
$$

(5)

follows by the stationarity conditions, where $e$ denotes the element number, $\phi^e_k$ is the restriction of $\phi_k$ to the domain of element $e$, $\{\phi_k\}$ is the finite element basis of $T^h$, $\xi^e_i$ is the $i$th integration point for element $e$, and $W^e_i$ is the weight associated with $\xi^e_i$. (If the change of variables formula is used before applying the integration rule, as in the isoparametric case, we assume $W^e_i$ subsumes the Jacobian determinant.)

Note that $\phi^e_k(\xi^e_i) = \delta_{kl}$ (the Kronecker delta) since the integration points $\xi^e_i$ correspond to the nodes of $T^h$. Using this fact in the above expression and assuming $W^e_i \neq 0$ implies

$$
0 = \frac{\partial \Psi}{\partial g}(\xi^e_i) = L(u_i)(\xi^e_i) - a g(\xi^e_i)
$$

(6)

for each integration point (node) $l$.

Let $\partial / \partial u_i$ mean "variation with respect to the $u_i$ variable"; then we also have

$$
0 = \int_{\Omega} \left[ \frac{\partial}{\partial u_i} [\text{rest}] + \frac{\partial \Psi}{\partial u_i} \right] \, d\Omega = \int_{\Omega} \left[ \frac{\partial}{\partial u_i} [\text{rest}] + g \frac{\partial L}{\partial u_i}(u_i) \right] \, d\Omega .
$$

(7)

Therefore, using numerical integration and substituting (6) into (7) at the integration points show that the single-field principle is stationary.

Conversely, stationarizing the single-field principle and defining $g$ as in 4') make (6) and (7) valid, implying that $(u_i, g)$ is the solution.

The following examples illustrate corresponding functionals for the mixed and single-field cases.

Example 5. Herrmann:

$$
\int_{\Omega} \left[ e_{ij} e_{ij} + 2\nu e_{ij} H - \nu(1 - 2\nu) H^2 \right] \, d\Omega
$$
becomes the usual compressible potential energy

\[
\int_{\Omega} \left[ \frac{\nu}{1 - 2\nu} (\varepsilon_{ij})^2 + \varepsilon_{ij}\varepsilon_{ij} \right] \, d\Omega,
\]

where summation of repeated indices is implied.

**Example 6. Beam:**

\[
\int_{0}^{L} \left[ \theta_{x}^2 + 2q(w_{x} - \theta) - \frac{t^2(1 + \nu)}{6} q^2 \right] \, dx
\]

becomes

\[
\int_{0}^{L} \left[ \theta_{x}^2 + \frac{6}{t^2(1 + \nu)} (w_{x} - \theta)^2 \right] \, dx.
\]

**Example 7. Plate:**

\[
\int_{A} \left\{ \theta_{1,1}^2 + 2\nu\theta_{1,1}\theta_{2,2} + \theta_{2,2}^2 + \frac{(1 - \nu)}{2} (\theta_{1,2} + \theta_{2,1})^2 \\
+ 2q_{1}(w_{1} - \theta_{1}) + 2q_{2}(w_{2} - \theta_{2}) - \frac{t^2}{6(1 - \nu)} (q_{1}^2 + q_{2}^2) \right\} \, dA
\]

becomes

\[
\int_{A} \left\{ \theta_{1,1}^2 + 2\nu\theta_{1,1}\theta_{2,2} + \theta_{2,2}^2 + \frac{(1 - \nu)}{2} (\theta_{1,2} + \theta_{2,1})^2 + \frac{6(1 - \nu)}{t^2} [(w_{1} - \theta_{1})^2 + (w_{2} - \theta_{2})^2] \right\} \, dA.
\]

**Example 8. Laplacian:**

\[
\int_{\Omega} [u_{i}v_{i} - \frac{1}{2} \varepsilon_{ij}v_{i}] \, d\Omega
\]

becomes

\[
\frac{1}{2} \int_{\Omega} u_{i}u_{i} \, d\Omega.
\]

**Remark 5.** A general rule is to choose \( T^h \) such that \( L(u_i) \notin T^h \). In methods which become Lagrange multiplier methods at \( \alpha = 0 \), such as examples 1–3, specific arguments can be given which show that for small \( \alpha \) Lagrange elements integrated with reduced integration formulas give
Remark 6. The appearance of $1/\alpha$ in $\Psi$ can be interpreted as a penalty, making constrained problems into a "penalty method" as $\alpha \to 0$ in the single-field version [25].

Remark 7. When $\Psi$ is only part of the single-variable principle, we call the application of reduced integration to $\Psi$ "selective". If the whole integrand is evaluated by the same formula, we call the application "uniform".

Remark 8. A sufficiently accurate integration rule on the $\epsilon_{ij} \epsilon_{ij}$ term in example 1 guarantees that under all physically appropriate cases the stiffness matrix is positive definite (this follows from a Korn inequality; see for example [26]). Unfortunately, in the case of plates no such analogous result holds, and zero-energy modes may be engendered by reduced integration of the shear terms (see [20]).

The following examples illustrate some reduced/selective integration rules for the single-field case and corresponding finite element spaces in the sense of construction (A) for the mixed case.

Example 9.

For definiteness, consider the case of 2-dimensional, plane strain elasticity theory. Mixed and single-field functionals are given in example 5. Assume that 4-node quadrilaterals are employed and that the displacement field is to have bilinear variation in the natural coordinates over each element. This defines $S^h$. Furthermore, assume that, in the single-field case, 1-point Gauss integration is used on the volume term, and $2 \times 2$ Gauss integration is used on the remaining term.

In the mixed case $T^h$ is then defined to be the space of constants over each element with nodes at the element centroids.

Example 10.

Again take the case of plain strain elasticity. This time assume that 9-node Lagrangian quadrilaterals are employed and that displacements have biquadratic variation over each element. This defines $S^h$. Assume that the single-field integration rule is $2 \times 2$ Gauss on the volume terms and $3 \times 3$ Gauss on the remaining term. The nodal points for $T^h$ are the $2 \times 2$ Gauss points, and $T^h$ consists of functions having bilinear variation over each element. These functions are, in general, discontinuous across element boundaries.

2.2. Equivalence of the matrix equations

The matrix equations of the mixed method are

$$K'u + \bar{K}g = F,$$

$$\bar{K}^t u - \alpha Qg = 0,$$

where $\bar{K}$ is the matrix from the $gL(u_i)$ terms, $Q$ is the matrix from the $g^2$ terms, $F$ is the vector of applied loads, and $K'$ is the matrix from the rest of the functional; Eq. (9) may be solved for $g$. 

and $g$ substituted in (8) to obtain

$$K'u + \frac{1}{\alpha} \bar{K}Q^{-1} \bar{K}^t u = F. \quad (10)$$

The matrix equation of the single-field principle is

$$K'u + \frac{1}{\alpha} Ku = F, \quad (11)$$

where $K$ is the matrix of the $L(u_i)^2$ terms.

The equivalence of (10) and (11) is established in the following:

**Theorem 2:** Construct $T^h$ as in (A) and use the integration formula/nodes as in the previous theorem. Then $K = \bar{K}Q^{-1} \bar{K}^t$.

**Proof.** Let $u_i$, $v_i \in S^h$ be arbitrary. Define $g(\xi_i^e)$ from $u_i$ by (6). Then the equivalent matrix equation (9) also holds. Therefore,

$$v^t Ku = \sum_e \sum_l W_i^e [L(v_l) L(u_i)] (\xi_i^e)$$

$$= \alpha \sum_e \sum_l W_i^e [L(v_l) g] (\xi_i^e)$$

$$= \alpha v^t \bar{K}g$$

$$= v^t \bar{K}Q^{-1} \bar{K}^t u.$$

The first and third lines are just the definitions of $K$ and $\bar{K}$, respectively.

**Remark 9.** $K'$ may be 0, as in example 8.

### 2.3. Applications to constrained media problems

The present ideas may be applied fruitfully to problems involving constraints. For example, as $\nu \to \frac{1}{2}$, the deformation field of an elastic continuum becomes isochoric, and as $t \to 0$ (the bending rigidity being held constant); the shear deformations in plates and beams approach zero etc. In such situations standard single-field finite element formulations have often behaved poorly, whereas mixed formulations have been effective. The reduced/selective integration concept achieves the effectiveness of the latter approach while maintaining the computational simplicity and efficiency of the former.

We note that in problems involving constraints a certain number of degrees of freedom are occupied in maintaining the constraints; the remainder represent the approximation capability of the finite element space. To assess the relative worth of various finite elements in problems of this sort, consider, for example, a 2-dimensional square mesh in which all degrees of freedom on two
Table 1. Elasticity elements, integration formulas and constraint indices in 2 dimensions
(d.p. = degree of precision)

<table>
<thead>
<tr>
<th>$S^h$ element type</th>
<th>Integration formula</th>
<th>$T^h$ element type</th>
<th>$N_{udof}/N_{es}^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>bilinear Lagrange</td>
<td>$1 \times 1$ Gauss</td>
<td>constant</td>
<td>1</td>
</tr>
<tr>
<td>linear triangle</td>
<td>1 pt. (d.p. = 1)</td>
<td>constant</td>
<td>0</td>
</tr>
<tr>
<td>biquadratic Lagrange</td>
<td>$2 \times 2$ Gauss</td>
<td>bilinear Lagrange</td>
<td>4</td>
</tr>
<tr>
<td>quadratic triangle</td>
<td>3 pt. (d.p. = 1)</td>
<td>linear triangle</td>
<td>2</td>
</tr>
<tr>
<td>quadratic serendipity</td>
<td>$2 \times 2$ Gauss</td>
<td>bilinear Lagrange</td>
<td>2</td>
</tr>
<tr>
<td>bicubic Lagrange</td>
<td>$3 \times 3$ Gauss</td>
<td>biquadratic Lagrange</td>
<td>9</td>
</tr>
<tr>
<td>cubic triangle</td>
<td>6 pt. (d.p. = 4)</td>
<td>quadratic triangle</td>
<td>6</td>
</tr>
<tr>
<td>cubic serendipity</td>
<td>$3 \times 3$ Gauss</td>
<td>biquadratic Lagrange</td>
<td>1</td>
</tr>
</tbody>
</table>

Mesh types:

- Rectangular
- Triangular

adjacent boundaries are fixed, the two opposite edges being free. The constraint index $N_{udof}/N_{es}^2$ — where $N_{udof}$ denotes the number of unconstrained degrees of freedom (i.e. the total number of degrees of freedom minus the number of constraints), and $N_{es}$ is the number of elements per side — is a constant for such a mesh and is a measure of the approximation capability of the element under consideration. The number of constraints per element may be counted by applying techniques described in [10–12].

Analogs in 1 and 3 dimensions are easily constructed. The greater the constraint index, the better the element; an index < 0 implies the element is overconstrained and will yield worthless numerical results in constrained applications.

In tables 1–4 we present data of this kind for typical finite element spaces for beam, plate and incompressible elasticity applications. These data indicate that the most desirable elements of those considered, in constrained media applications are the so-called Lagrange elements with appropriate reduced/selective integration formulas. Serendipity elements come off rather poorly. It has been our numerical experience that there is a close correlation between constraint indices and the relative worth of the various finite elements in constrained media applications.

Table 2. Elasticity elements, integration formulas and constraint indices in 3 dimensions
(mesh types are 3 dimensional generalizations of those in table 1)

<table>
<thead>
<tr>
<th>$S^h$ element type</th>
<th>Integration formula</th>
<th>$T^h$ element type</th>
<th>$N_{udof}/N_{es}^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>trilinear Lagrange</td>
<td>$1 \times 1 \times 1$ Gauss</td>
<td>constant</td>
<td>2</td>
</tr>
<tr>
<td>linear tetrahedron</td>
<td>1 pt. (d.p. = 1)</td>
<td>constant</td>
<td>-2</td>
</tr>
<tr>
<td>triquadratic Lagrange</td>
<td>$2 \times 2 \times 2$ Gauss</td>
<td>trilinear Lagrange</td>
<td>16</td>
</tr>
<tr>
<td>quadratic tetrahedron</td>
<td>4 pt. (d.p. = 1)</td>
<td>linear tetrahedron</td>
<td>4</td>
</tr>
<tr>
<td>quadratic serendipity</td>
<td>$2 \times 2 \times 2$ Gauss</td>
<td>trilinear Lagrange</td>
<td>54</td>
</tr>
<tr>
<td>tricubic Lagrange</td>
<td>$3 \times 3 \times 3$ Gauss</td>
<td>quadratic tetrahedron</td>
<td>19</td>
</tr>
<tr>
<td>cubic tetrahedron</td>
<td>10 pt. (d.p. = 4)</td>
<td>quadratic tetrahedron</td>
<td>-9</td>
</tr>
<tr>
<td>cubic serendipity</td>
<td>$3 \times 3 \times 3$ Gauss</td>
<td>biquadratic Lagrange</td>
<td>1</td>
</tr>
</tbody>
</table>
We note that the reduced/selective integration approach applied to very thin beams and plates may be viewed as a penalty function formulation of the so-called "discrete Kirchhoff hypothesis", the points of zero shear strains being the integration points.

Similarly, the pure displacement method for incompressibility problems is a penalty method with penalty $z = \nu / (1 - 2\nu)$ enforcing near-zero volume strain for $\nu$ near $\frac{1}{2}$. As was pointed out earlier, reduced/selective integration is essential in obtaining accurate solutions for large $z$.

2.4. Numerical examples

Example 11. Nearly incompressible elasticity:

Consider the equations of two-dimensional linear isotropic elasticity theory for the domain illustrated in fig. 1. The boundary conditions are given as follows:

(displacement)

$$u_1(0, 0) = u_2(0, 0) = 0,$$

$$u_1(0, \pm c) = 0.$$

(traction)

$$t_1(x_1, \pm c) = t_2(x_1, \pm c) = 0, \quad x_1 \in (0, L),$$

$$t_1(L, x_2) = 0$$

$$t_2(L, x_2) = \frac{p}{2L} (c^2 - x_2^2) \quad x_2 \in (-c, c).$$
where $P$ is a given constant and $I = 2c^3/3$.

The traction boundary conditions are those encountered in simple bending theory for a cantilever beam with root section at $x = 0$, that is parabolically varying end shear and linearly varying bending stress at the root. The displacement boundary conditions allow the root section to warp.

The following data were employed in the numerical calculations:

$$L = 16, \ c = 2.$$
Vertical tip displacements [i.e., $u_2(16, 0)$] are compared in table 5. Both the "standard" $2 \times 2$ Gauss quadrature element and the selective/reduced element provide adequate results for $\nu = 0.3$. However, for the nearly-incompressible case the standard quadrilateral degenerates, whereas the selective/reduced element retains accuracy. The latter element is identical to the constant pressure — bilinear displacements mixed model based upon Herrmann's formulation [3]. Higher-order quadrilateral and triangular elements are considered in [13].

Example 12. Beams:

The simplest beam element which can be devised is constructed by assuming linear functions for both $w$ and $\theta$. One-point Gauss integration of the shear term results in an element identical to a mixed element in which $\theta$ is assumed constant, and $w$ and $\theta$ are linear. Numerical calculations were performed with this model for thick and thin beams. The data used are as follows:

(thick beam) \hspace{1cm} L = 4 \hspace{1cm} t = 0.79 \hspace{1cm} \nu = 0.3

(thin beam) \hspace{1cm} L = 4 \hspace{1cm} t = 2.5 \times 10^{-3} \hspace{1cm} \nu = 0.3

Tip displacement results for a cantilever beam are presented in tables 6 and 7. In each case the 1-point Gauss integration element is superior to the 2-point Gauss element, the differences being particularly striking for the thin beam. For a more complete discussion of this element see [20].

<table>
<thead>
<tr>
<th>Number of elements</th>
<th>1-point Gauss</th>
<th>2-point Gauss</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thick beam</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>.762</td>
<td>.416 $\times 10^{-1}$</td>
</tr>
<tr>
<td>2</td>
<td>.940</td>
<td>.445</td>
</tr>
<tr>
<td>4</td>
<td>.985</td>
<td>.762</td>
</tr>
<tr>
<td>8</td>
<td>.996</td>
<td>.927</td>
</tr>
<tr>
<td>16</td>
<td>.999</td>
<td>.981</td>
</tr>
</tbody>
</table>

| Thin beam          |               |               |
| 1                  | .750          | .200 $\times 10^{-4}$ |
| 2                  | .938          | .800 $\times 10^{-4}$ |
| 4                  | .984          | .320 $\times 10^{-3}$ |
| 8                  | .996          | .798 $\times 10^{-3}$ |
| 16                 | .999          | .512 $\times 10^{-3}$ |

The "standard" beam element including shear deformation (see [27] p. 333) can be derived from the reduced integration concept as follows: Assume that both $w$ and $\theta$ vary quadratically within an element. Use 2-point integration on all terms and statically condense the degrees of freedom associated with the middle node. (We thank H. Allik for pointing this out to us.)

Example 13. Plates:

The simplest effective plate element based upon the reduced and selective integration concept
consists of assuming that $w$ and $\theta_{\alpha}$ are bilinear functions of the natural coordinates on a quadrilateral. One-point Gauss integration on the shear terms results in an element identical to the mixed model in which $q_{\alpha}$ is assumed constant, and $w$ and $\theta_{\alpha}$ are bilinear. Despite the simplicity of this element it appears to be competitively accurate when compared to more involved elements (see [20]). Numerical results for the mesh depicted in fig. 3 and refinements obtained by bisection are contained in table 8. The data employed were

$$L = 10, \quad t = 10^{-5/2}, \quad \nu = 0.3.$$ 

![Fig. 3. Square plate (due to symmetry only one quadrant is discretized).](image)

Two-by-two Gauss integration of the shear term, as in the case for the beam, leads to worthless numerical results for a plate this thin.

Table 7. Normalized center displacement and bending moment for a simply supported square plate

<table>
<thead>
<tr>
<th>Number of elements</th>
<th>Displacement – concentrated load</th>
<th>Displacement – uniform load</th>
<th>Moment – uniform load</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>.9922</td>
<td>.9770</td>
<td>.851</td>
</tr>
<tr>
<td>16</td>
<td>.9948</td>
<td>.9947</td>
<td>.963</td>
</tr>
<tr>
<td>64</td>
<td>.9982</td>
<td>.9987</td>
<td>.991</td>
</tr>
</tbody>
</table>

3. Nonlinear analysis

3.1. Equivalence of the solutions

In (1) let $L(u_i)$ be replaced by $N(u_i)$, where $N$ is now a nonlinear operator (possibly tensor valued). The same relation exists between the Euler equation and $\Psi$ (see (2) and (3)), except now $\Psi$ is not quadratic in $u_i$. Theorem 1 does not rely on the linearity of $L$, now $N$, or the quadratic nature of $\Psi$, only stationarity. The proof holds in the nonlinear case, showing that any stationary point of the single-field principle gives a stationary point of the mixed principle by adjoining $\{g\} = \{\alpha^{-1}N(u_i)\ (\xi_i)\}$. The converse also holds, and there may be more than one stationary point.

Example 14. Generalization of Herrmann's principle to nonlinear elasticity:

In this case we have
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\[ N(u_i) = G(\text{III}(u_i) - 1) , \]

where \( \text{III} \) is the third invariant of the Green deformation tensor (\( \text{III} = 1 \) at constant volume), and \( G(x) = 0 \) if and only if \( x = 0 \). The functional for the generalized Herrmann's principle is

\[
\text{(a)} \quad \int_{\Omega} \left[ F(I, \text{II}, \text{III}) + \alpha G(\text{III} - 1) - \frac{\alpha \sigma^2}{2} \right] \, d\Omega
\]

where \( I \) and \( \text{II} \) are the first and second invariants of the Green deformation tensor, respectively, and \( \alpha > 0 \) is a parameter which tends to zero as the compressibility tends to zero. At \( \alpha = 0 \) the formulation becomes a Lagrange multiplier method, with multiplier \( g \), enforcing incompressibility (i.e. \( \text{III} = 1 \)).

The corresponding single-field functional is

\[
\text{(b)} \quad \int_{\Omega} \left[ F(I, \text{II}, \text{III}) + \frac{\alpha^{-1}}{2} \text{G}^2 (\text{III} - 1) \right] \, d\Omega .
\]

**Remark 9.** For \( \alpha > 0 \) the solutions for (a) and (b) are identical. Under reasonable hypotheses \( \alpha^{-1} G(\text{III} - 1) \) exists in the limit \( \alpha \to 0 \) and is equal to the "g" obtained by setting \( \alpha = 0 \) in (a) (see Luenberger [25]).

**Remark 10.** In the usual formulation of incompressible isotropic nonlinear elasticity \( F(I, \text{II}, \text{III}) = F(I, \text{II}) \) and \( G(\text{III} - 1) = \text{III} - 1 \) (see Oden [28]). In this case \( g \) turns out to be the hydrostatic pressure, or isotropic component of the Cauchy stress tensor. For more general \( G(x) \) we have

\[ G(x) = x G'(0) + \frac{x^2}{2} G''(r), \quad r \in [0, x] . \]

Near constant volume \( G(\text{III} - 1) \approx (\text{III} - 1) G'(0) \), and so at \( \text{III} = 1 \) our "g" differs from the hydrostatic pressure by a factor of \( G'(0) \) (see [17] for an alternative formulation of nonlinear elasticity).

**Remark 11.** Nonlinear analogs of the beam and plate example of section 2 are also easily constructed.

### 3.2. Equivalence of the linearized matrix equations

Consider the following prototypical pair of functionals:

**(mixed)**

\[
\int_{\Omega} \left[ \mathcal{W} + g N + \frac{\alpha}{2} \sigma^2 \right] \, d\Omega .
\]

**(single-field)**

\[
\int_{\Omega} \left[ \mathcal{W} + \frac{\alpha^{-1}}{2} N^2 \right] \, d\Omega .
\]
Linearized equations, which are employed in iterative/incremental solution strategies, are obtained by taking second variations of the functionals. The coefficient matrices in these equations are called hessians and for the above cases are given as follows:

**(mixed)**

\[
\begin{bmatrix}
  K_2 + \alpha^{-1} \bar{K}_0 & \bar{K}_1 \\
  \bar{K}_1^T & -\alpha Q
\end{bmatrix},
\]

where

\[
K_2 = \int_{\Omega} \frac{\partial^2 W}{\partial u \partial u} \, d\Omega, \quad \bar{K}_0 = \int_{\Omega} \alpha g \frac{\partial^2 N}{\partial u \partial u} \, d\Omega, \quad \bar{K}_1 = \int_{\Omega} \frac{\partial N}{\partial u} \left( \frac{\partial g}{\partial g} \right)^T \, d\Omega, \quad Q = \int_{\Omega} \frac{\partial g}{\partial g} \left( \frac{\partial g}{\partial g} \right)^T \, d\Omega.
\]

**(single field)**

\[
K_2 + \alpha^{-1} (K_1 + K_0),
\]

where

\[
K_0 = \int_{\Omega} N \frac{\partial^2 N}{\partial u \partial u} \, d\Omega, \quad K_1 = \int_{\Omega} \frac{\partial N}{\partial u} \left( \frac{\partial N}{\partial u} \right)^T \, d\Omega, \quad K_2 = \int_{\Omega} \frac{\partial^2 W}{\partial u \partial u} \, d\Omega.
\]

The (incremental) \( g \) variables may be eliminated from the linearized equations in the mixed case to yield the reduced hessian:

\[
K_2 + \frac{1}{\alpha} (\bar{K}_0 + \bar{K}_1 Q^{-1} \bar{K}_1^T).
\]

The following theorem establishes the relation between this matrix and the single-field hessian.

**Theorem 3.** Construct \( T^h \) as in (A) and use the integration formula/nodes as in theorem 1. Then

\[
K_1 = \bar{K}_1 Q^{-1} \bar{K}_1^T
\]

and

\[
\frac{1}{\alpha} (K_0 - \bar{K}_0) = \sum_e \sum_i W_i^e \left[ \frac{1}{\alpha} N(\xi_i^e) - g(\xi_i^e) \right] \frac{\partial^2 N}{\partial u \partial u}(\xi_i^e).
\]

**Proof.** The first result may be proved by a calculation identical to the one used in theorem 2. The second result is an immediate consequence of the definitions of \( K_0 \) and \( \bar{K}_0 \).

**Corollary.** At a stationary point for the mixed principle the reduced hessian is identical to the single-field hessian.

**Proof.** At a stationary point, \( g(\xi_i^e) = \alpha^{-1} N(\xi_i^e) \). Thus by theorem 3, \( K_0 = \bar{K}_0 \), and the conclusion follows.
Remark 12. By continuity, near a stationary point $\alpha^{-1}N(\xi^e) - g(\xi^e)$ will be small, and thus the hessians will differ by a perturbation given explicitly by the second result of theorem 3.

3.3. Application

Example 15.

The reduced/selective integration technique can be used successfully with a penalty function formulation of the Navier-Stokes equations for an incompressible fluid. For nonzero Reynolds number there is no variational principle from which to derive the appropriate equations. However, as is now well known, this is no hindrance as a Galerkin approach may be adopted instead.

To simplify the present exposition, assume homogeneous boundary conditions. The Galerkin equation for the usual pressure-velocity (mixed) formulation is given as follows: Find $p \in T^h$ and $\psi \in S^h$ such that for all $q \in T^h$ and $\psi \in S^h$

$$0 = \int_{\Omega} \left\{ \psi_i f_i - \rho \psi_i v_{i,j} + p \psi_{i,i} - 2 \mu \psi_{(i,j)} v_{(i,j)} + q v_{i,j} \right\} \, d\Omega,$$

where $f_i$ is the prescribed body force, $\rho$ is the density, $\mu$ is the viscosity, and $v_{(i,j)} = (v_{i,j} + v_{j,i})/2$.

The corresponding penalty function (single-field) equation is

$$0 = \int_{\Omega} \left\{ \psi_i f_i - \rho \psi_i v_{i,j} - \lambda \psi_{i,i} v_{i,j} - 2 \mu \psi_{(i,j)} v_{(i,j)} \right\} \, d\Omega,$$

where $\lambda$ is the penalty function parameter. If $\lambda$ is taken sufficiently large, velocities based upon the latter formulation closely approximate those of the former. The pressures in the latter case are given by $-\lambda v_{i,i}$.

The choice of elements and integration rules for the problem under consideration is identical to that for elasticity (see tables 1 and 2). The simplest effective scheme in 2-dimensional calculations employs the 4-node quadrilateral in which bilinear shape functions are used for the velocities. In the Galerkin equation, $2 \times 2$ Gauss quadrature is used on all terms except the $\lambda$ term, for which the 1-point Gauss rule is used.

Numerical calculations were performed with this model for the mesh depicted in fig. 4. A uniform radial velocity of magnitude $1/4$ was specified at $r = 4$ pointing towards the vertex. The inner radius ($r = 1/4$) was assumed traction-free. The data used are given as follows:

$$\rho = 10,800, \quad \mu = 1, \quad \lambda = 10^9.$$

The resulting Reynolds number is 5,655 (see Batchelor [29] p. 295). In figs. 5 and 6 comparisons of computed data with the Hamel solution (Batchelor [29] pp. 294–302) are given. The correlation is seen to be very good. For further details concerning the present approach the interested reader may consult [18, 19].
Fig. 4. Element mesh for convergent flow in a channel.

Fig. 5. Pressure profile for convergent flow in a channel.
4. Summary and conclusions

General theorems have been stated and proved which establish the equivalence of certain mixed finite element methods with displacement methods employing the reduced/selective integration concept in both linear and nonlinear problems.

Applications and numerical examples have been given for beams, plates and incompressible and nearly-incompressible problems in fluid and solid mechanics.

The significance of these results is that the accuracy of the mixed methods can be attained by the reduced/selective integration displacement method without engendering the computational expense of the former, and the now highly developed mathematical theory of the mixed methods immediately applies to the reduced/selective integration elements.

It has also been pointed out that in constrained problems the reduced/selective integration technique is equivalent to a penalty function formulation. In the case of beams and plates this turns out to be the "discrete Kirchhoff hypothesis". For incompressible elasticity problems it turns out to be the compressible formulation with a penalty on the volume strain.

Of the commonly used higher-order element families only the Lagrange family is effective for constrained media situations.

Several topics deserve further study. An important one is the zero-energy modes of plates. Whether or not these are globally troublesome is an open question (see [20]).

It would also be worthwhile to ascertain the relationship, if any, of the incompatible modes concept [8, 30] to the hybrid methods (see for example [31]). There is some evidence that a relationship exists (see Gallagher [32]).

The present ideas will no doubt prove fruitful in many related areas.

References


